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LETTER TO THE EDITOR

The Darboux–Bäcklund transformation without using a matrix representation

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Abstract. We present a new method to construct soliton solutions for a large class of nonlinear systems which are associated with matrix spectral problems of high degree (including a reduction of the Lamé equations and a generalized sine–Gordon system). The spectral problems are represented in terms of Clifford numbers. Then the solutions can be obtained in a straightforward way without using any matrix representation. Among the spectral problems under consideration is the Dirac equation for a massless particle in the electromagnetic field and its generalizations.

Integrable systems are usually associated with scalar spectral problems (such as the Korteweg–de Vries hierarchy, associated with the stationary Schrödinger operator [1]) or matrix spectral problems of low degree. The classical example is given by the AKNS class of integrable systems (including the nonlinear Schrödinger equation, modified Korteweg–de Vries equation and sine–Gordon equation) based on $\boldsymbol{sl}(2, \boldsymbol{C})$ spectral problems [2]. The $\boldsymbol{su}(2)$ reduction of the AKNS class is associated with the $1+1$ analogue of the Dirac equation. The matrix spectral problems of higher degree (in practice greater than three) create, as a rule, serious technical problems and, in general, not many examples are known.

In this letter we extend one of the standard techniques of the soliton theory, namely the Darboux–Bäcklund transformation, to a class of spectral problems in Clifford algebras. They are formally identical with matrix spectral problems $\partial\Psi/\partial x^k = U_k\Psi$ $(k = 1, \ldots, n)$ of high degree but $U_1, \ldots, U_n$ and Ψ assume values in a given Clifford algebra. Representing Clifford numbers by matrices (this can always be done [3]) we may return to a matrix spectral problem. Well known examples of generators of Clifford algebra are Pauli and Dirac matrices. It turns out, however, that it is possible and more convenient to work directly with Clifford numbers.

Let us begin with two examples associated with the Clifford algebra $\mathcal{C}(2n)$. First, consider the linear problem [4]

$$\Psi_{,k} = \tfrac{1}{2}\gamma_k(\lambda \boldsymbol{a}_k + \boldsymbol{b}_k)\Psi \qquad (k = 1, \ldots, n) \tag{1}$$

where γ_μ $(\mu = 1, \ldots, 2n)$ generate the Clifford algebra $\mathcal{C}(2n)$, i.e. they satisfy relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$; then $\boldsymbol{a}_k := \sum_{j=1}^{n} \alpha_{kj}\gamma_{n+j}$, $\boldsymbol{b}_k := \sum_{j=1}^{n} \beta_{kj}\gamma_j$ and α_{kj}, β_{kj} are some scalar functions of $x^1, \ldots, x^n$. We assume $\beta_{kk} = 0$. The compatibility conditons for that system read as follows:

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_{ji}\alpha_{ki} = 0 \qquad \alpha_{ji},_k + \alpha_{ki}\beta_{jk} = 0 \qquad (j \neq k) \\ &\beta_{ji},_k + \beta_{jk}\beta_{ki} = 0 \qquad (i \neq j \neq k \neq i) \\ &\beta_{jk},_k + \beta_{kj},_j = \sum_{i=1}^{n} \beta_{ji}\beta_{ki} \qquad (j \neq k). \end{aligned} \tag{2}$$

From the first two equations it follows that $\sum_{i=1}^n \alpha_{ji},_k \alpha_{ji} = 0$, so we assume that $\sum_{i=1}^n \alpha_{ji}^2$ does not depend on $x^1, \ldots, x^n$. For a fixed index j, substituting $\alpha_{kj} =: H_k$ and $\beta_{kj} = -\alpha_{ji}^{-1}\alpha_{ki},_j = H_j^{-1}H_k,_j$, we obtain the Lamé equations in the classical form but with the following additional constraint:

$$\sum_{i=1}^{n} H_i^2 = \text{constant}.$$

This system is sometimes called the 'generalized wave equation' [5] because in the case $n = 2$ it reduces to the wave equation $\vartheta,_{11} - \vartheta,_{22} = 0$ (where $\cos\vartheta := \alpha_{11}$). In the case $n = 3$ we obtain a system of six nonlinear partial differential equations for three unknowns, H_1, H_2 and H_3, with the algebraic constraint $H_1^2 + H_2^2 + H_3^2 = \text{constant}$.

The second example has a similar form:

$$\Psi,_k = \frac{1}{2}\gamma_k\left(\lambda \boldsymbol{a}_k - \frac{1}{\lambda}\gamma_{n+1}\boldsymbol{a}_k\gamma_{n+1} + \boldsymbol{b}_k\right)\Psi \qquad (k = 1, \ldots, n) \tag{3}$$

where γ_μ ($\mu = 1, \ldots, 2n$) are also generators of $\mathcal{C}(2n)$ and $\boldsymbol{a}_k := \sum_{j=1}^n \alpha_{kj}\gamma_{n+j}$, $\boldsymbol{b}_k := \sum_{j=1}^n \beta_{kj}\gamma_j$. Note that $\gamma_{n+1}\boldsymbol{a}_k\gamma_{n+1} = \alpha_{k1}\gamma_{n+1} - \sum_{j=2}^n \alpha_{kj}\gamma_{n+j}$. The compatibility conditions yield the so-called generalized sine–Gordon system [6]:

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_{ji}\alpha_{ki} = 0 \qquad \alpha_{ji},_k + \alpha_{ki}\beta_{jk} = 0 \qquad (j \neq k) \\ &\beta_{ji},_k + \beta_{jk}\beta_{ki} = 0 \qquad (i \neq j \neq k \neq i) \\ &\beta_{jk},_k + \beta_{kj},_j + \sum_{i=1}^{n} \beta_{ji}\beta_{ki} = \alpha_{1j}\alpha_{1k} \qquad (j \neq k) \end{aligned} \tag{4}$$

where indices i, j, k run from 1 to n.

Almost identical spectral problems can be found in the paper by Ablowitz *et al* [5], where instead of Clifford numbers $\boldsymbol{so}(n, n)$ matrices have been used. In both cases the wave function Ψ takes values in the group $\mathrm{Spin}(2n)$ (in a matrix representation the columns of Ψ can be identified with linearly independent spinor solutions). The simplest example of an application of the Spin group in the soliton theory is the AKNS class of integrable systems. Some of its members (e.g. the sine–Gordon equation) are associated in a natural way with the $\boldsymbol{SO}(3)$ group. However, from both technical and theoretical points of view it is better to use the covering group $\mathrm{Spin}(3) \simeq \boldsymbol{SU}(2)$ (compare [7]).

Both examples have a disadvantage (typical for spectral problems of high degree): the associated nonlinear systems involve a large number of equations and fields. The important point is to find some simple characterization of such systems. A natural possibility is to define geometrical objects which are in one-to-one correspondence with solutions to the system under consideration.

In this context it is worthwhile to mention the so-called 'soliton surface approach' ([8–10], compare also [11–13]), which associates the modern theory of solitons with the classical differential geometry in the spirit of Bianchi, Darboux and other great geometers of the 19th century [14].

Given a matrix wavefunction Ψ let us define F using the so-called Sym–Tafel formula $F = \Psi^{-1}\Psi,_\lambda$ [8]. The matrix F is an element of some linear space. If Ψ assumes values in a matrix Lie group, then $F = F(x, t; \lambda)$ is a λ-family of submanifolds immersed in the corresponding Lie algebra.

The case of $\Psi \in \boldsymbol{SU}(2)$ is of special importance. F describes a surface immersed in $\boldsymbol{su}(2)$ which is well known to be identified with the three-dimensional Euclidean space. Many applications in differential geometry and in physics have been found [9, 11, 15, 16]. Let us mention the localized induction equation describing the motion of a single vortex filament in an ideal fluid [17, 18]. The soliton surface approach gives a natural correspondence between this equation, the nonlinear Schrödinger equation and the one-dimensional continuum Heisenberg ferromagnet model [15, 17]. Another good example is given by the sine–Gordon equation $\varphi,_{xt} = \sin\varphi$. The corresponding 'soliton surfaces' are pseudospherical surfaces in $\boldsymbol{R}^3$, the radius vector solves a model of the relativistic string and the normal vector N solves the two-dimensional $\boldsymbol{O}(3)$-invariant σ-model: $N,_{tt} - N,_{xx} + (N,_t^2 - N,_x^2)N = 0$ [15].

The main advantage of the soliton surface approach consists in explicit formulas for the radius vector F. In other approaches (in the case of localized induction equation compare, e.g., [19]) one has first to solve a nonlinear system associated with a given model, and then to solve linear differential equations to reconstruct F. This second step, involving cumbersome calculations, is omitted from our approach.

In both examples discussed in this letter $F = \Psi^{-1}\Psi,_\lambda$ admits a nice interpretion. In the first case $F = \Psi^{-1}\Psi,_\lambda|_{\lambda=0}$ has the form

$$F = F^{(1)}\gamma_{n+1} + F^{(2)}\gamma_{n+2} + \cdots + F^{(n)}\gamma_{2n} \tag{5}$$

where $F^{(k)} := F_{k1}\gamma_1 + F_{k2}\gamma_2 + \cdots + F_{kn}\gamma_n$ for $k = 1, \ldots, n$ and F_{kj} are scalar functions of $x^1, \ldots, x^k$. One can prove that for any k the vector $F^{(k)}$ (considered as a function in $\boldsymbol{R}^n$, i.e. $F^{(k)} \longleftrightarrow [F_{k1}, \ldots, F_{kn}]$) describes an orthogonal net in $\boldsymbol{R}^n$. In other words, F describes a submanifold immersed in $\boldsymbol{R}^{n^2}$ (Cartesian product of n copies of $\boldsymbol{R}^n$) and projecting it on the spaces $\boldsymbol{R}^n$ we obtain n copies of orthogonal nets. In the second case, one can prove that F (evaluated at $\lambda = 1$) explicitly describes n-dimensional Lobachevsky spaces (i.e. spaces of constant negative curvature) immersed in Euclidean spaces of dimension $2n - 1$ [20].

The first example can be generalized as follows. Let V and W be orthogonal vector spaces ($\dim V = r$, $\dim W = q$) endowed with a non-degenerate (but not necessarily positive definite) scalar product. Consider spectral problems of the form [21]:

$$\Psi,_k = \tfrac{1}{2}\gamma_k(\lambda \boldsymbol{a}_k + \boldsymbol{b}_k)\Psi \qquad (k = 1, \ldots, n) \tag{6}$$

where $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ assume values in W and V respectively, $\gamma_1, \ldots, \gamma_r$ is an orthonormal basis in V and $\gamma_{r+1}, \ldots, \gamma_{r+q}$ is an orthogonal basis in W. Both vector spaces (V and W) are considered as subspaces in the Clifford algebra $\mathcal{C}(V \oplus W)$, i.e.

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\eta_{kj} \qquad (j, k = 1, \ldots, r + q) \tag{7}$$

($\eta_{kj} = 0$ for $k \neq j$, $\eta_{jj} = \pm 1$).

The main result of our letter is the construction of the Darboux–Bäcklund transformation for any spectral problem of the type (6). Let us postulate the Darboux operator (a natural analogue of the Darboux matrix) in the following, quite simple, form:

$$D = \boldsymbol{f}_0(\lambda \boldsymbol{n} + \kappa \boldsymbol{p}) \tag{8}$$

where $\boldsymbol{f}_0 \in W$ is a constant unit vector, λ and κ are real parameters, $\boldsymbol{n}$ is a W-valued function and, finally, $\boldsymbol{p}$ is V-valued. One can prove that $\boldsymbol{p}^2$ and $\boldsymbol{n}^2$ do not depend on $x^1, \ldots, x^n$. Here we confine ourselves to the case $\boldsymbol{p}^2 = \boldsymbol{n}^2$. We recall that the Darboux transformation is defined by $\tilde{\Psi} = D\Psi$. The corresponding transformations for $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ can be derived immediately:

$$\begin{aligned} \tilde{\boldsymbol{a}}_k &= \boldsymbol{f}_0 \boldsymbol{n} \boldsymbol{a}_k \boldsymbol{n}^{-1} \boldsymbol{f}_0 \\ \tilde{\boldsymbol{b}}_k &= \boldsymbol{b}_k + 2\kappa \langle \boldsymbol{n} | \boldsymbol{a}_k \rangle (\langle \boldsymbol{e}_k | \boldsymbol{p}^{-1} \rangle \boldsymbol{e}_k^{-1} - \boldsymbol{p}^{-1}). \end{aligned} \tag{9}$$

Our earlier results (cf [21, 22]) stopped at this point. The crucial problem consisted in finding an algebraic procedure to construct $\boldsymbol{n}$ and $\boldsymbol{p}$. The solution turned out to be surprisingly simple:

$$\boldsymbol{p} + \mathrm{i}\boldsymbol{n} = \Psi(\mathrm{i}\kappa)(\boldsymbol{p}_0 + \mathrm{i}\boldsymbol{n}_0)\Psi(\mathrm{i}\kappa)^{-1} \tag{10}$$

where $\boldsymbol{p}_0 \in V$ and $\boldsymbol{n}_0 \in W$ are constant. Note that the left-hand side has to be of the desired form (i.e. an element of $V \oplus \mathrm{i}W$) because $\Psi(\mathrm{i}\kappa)$ is an element of the corresponding group Spin and therefore the conjugation by $\Psi(\mathrm{i}\kappa)$ leaves the space $V \oplus \mathrm{i}W$ invariant.

Are the presented considerations effective? The answer is positive. In spite of the rather sophisticated setting, one can construct soliton solutions using straightforward, quite elementary, computations. The algorithm for finding soliton solutions consists of the following steps.

(1) Find a special solution $\Psi = \Psi(\lambda)$ (usual starting point of any Darboux–Bäcklund transformation).
(2) Choose a real parameter κ and compute $\Psi(\mathrm{i}\kappa)$.
(3) Choose constant Clifford vectors: $\boldsymbol{p}_0 \in V$, $\boldsymbol{n}_0 \in W$ and $\boldsymbol{f}_0 \in W$.
(4) Compute $\Psi(\mathrm{i}\kappa)(\boldsymbol{p}_0 + \mathrm{i}\boldsymbol{n}_0)\Psi(\mathrm{i}\kappa)^{-1}$. The result is a vector of the form $\boldsymbol{p} + \mathrm{i}\boldsymbol{n}$ where $\boldsymbol{p} = \boldsymbol{p}(x^1, \dots, x^n) \in V$ and $\boldsymbol{n} = \boldsymbol{n}(x^1, \dots, x^n) \in W$.
(5) Compute $\tilde{\Psi}(\lambda) := \boldsymbol{f}_0(\lambda\boldsymbol{n} + \kappa\boldsymbol{p})\Psi(\lambda)$, which is (automatically!) a new solution to the considered spectral problem.

The corresponding transformations for F and $\tilde{F}^{(k)}$ have also a compact form:

$$\begin{aligned} \tilde{F} &= F + \kappa^{-1}\Psi(0)^{-1}\boldsymbol{p}^{-1}\boldsymbol{n}\Psi(0) \\ \tilde{F}^{(k)} &= F^{(k)} + \nu_k\kappa^{-1}\Psi(0)^{-1}\boldsymbol{p}^{-1}\Psi(0) \end{aligned}$$

where ν_k are components of $\boldsymbol{n}$ (i.e. $\boldsymbol{n} = \sum_{k=1}^{q} \nu_k \gamma_{r+k}$). Note that we add to the given $F^{(k)}$ a segment (vector) of a constant length. This is a typical property of standard Bäcklund transformations (e.g. the transformation for the sine–Gordon equation and pseudospherical surfaces).

The identical procedure of constructing soliton solutions can be applied to a more general class of spectral problems as well:

$$\Psi_{,k} = \gamma_k \boldsymbol{u}_k \Psi \qquad (k = 1, \dots, n) \tag{11}$$

where γ_k generate a Clifford algebra defined by (7) and $\boldsymbol{u}_k$ are linear combinations of γ_k with (scalar) coefficients depending on $x^1, \dots, x^n$ and λ. We assume that $\boldsymbol{u}_k = \boldsymbol{u}_{k1} + \boldsymbol{u}_{k2}$, where $\boldsymbol{u}_{k1}$ assume values in V, $\boldsymbol{u}_{k2}$ assume values in W and, finally, $\boldsymbol{u}_{k1}(-\lambda) = \boldsymbol{u}_{k1}(\lambda)$, $\boldsymbol{u}_{k2}(-\lambda) = -\boldsymbol{u}_{k2}(\lambda)$. All additional constraints needs separate considerations. For instance, in the case of the spectral problem (3) the coefficients by λ are closely related to coefficients by $1/\lambda$. Indeed, we have $\boldsymbol{u}_k(1/\lambda) = -\gamma_{n+1}\boldsymbol{u}_k(\lambda)\gamma_{n+1}$.

The range of applications of the nonlinear systems associated with orthogonal coordinates (usually they are systems of hydrodynamic type [23]) varies from classical mechanics [24] to the quantum field theory [25].

An apparently novel application of the presented spectral problems is their close connection with the Dirac equation for a massless particle in the electromagnetic field. Consider the spectral problem (6). Let assume $r = n = 4$ and $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -\gamma_4^2 = -1$. Nothing is assumed about q. We assume that $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ are imaginary, i.e. $\boldsymbol{a}_k = -\mathrm{i}\sum_{j=5}^{4+q} a_{kj}\gamma_j$ and

$\boldsymbol{b}_k = -\mathrm{i}\sum_{j=1}^4 b_{kj}\gamma_j$ (a_{kj} and b_{kj} being real functions). Multiplying every equation (6) from the left by $\mathrm{i}\gamma_k^{-1}$, adding them together and changing the notation, $\gamma_k^{-1} \to \gamma^k$ ($k = 1, 2, 3$), $\gamma_4^{-1} \to \gamma^0$, we obtain

$$\sum_{\mu=0}^{3} \gamma^\mu(\mathrm{i}\partial_\mu - A_\mu)\Psi = \lambda C\Psi$$

where A_μ ($\mu = 0, 1, 2, 3$) are (real) coefficients by γ^μ in $\frac{\mathrm{i}}{2}\sum_{\nu=1}^4 \boldsymbol{b}_\nu$ and $C := \frac{\mathrm{i}}{2}\sum_{\nu=1}^4 \boldsymbol{a}_\nu$. For $\lambda = 0$ we recognize the Dirac equation for a massless particle in the electromagnetic field. Its solution Ψ has to satisfy the remaining three independent equations of the system (6) as well. Therefore we can generate in this way just a special class of solutions to the Dirac equation.

The equations (9) do not change if $\boldsymbol{a}_k \to \mathrm{i}\boldsymbol{a}_k$ and $\boldsymbol{b}_k \to \mathrm{i}\boldsymbol{b}_k$. Therefore the presented construction of the Darboux–Bäcklund transformation is valid for the Dirac equation case as well.

This rather unexpected relation of the Dirac equation to the integrable system of Lamé-type equations gives a clear possibility of constructing new exact solutions of the Dirac equation.

In conclusion let us note that the class of spectral problems considered in this letter is associated with the 3+1 Dirac equation (and its higher-dimensional analogues) in a similar way as the KdV hierarchy is associated with the Schrödinger operator and the modKdV hierarchy with the 1 + 1 Dirac operator. We stress that in both cases Spin groups have a fundamental role in the construction of the Darboux–Bäcklund transformation.

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References

[1] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg–de Vries equation *Phys. Rev. Lett.* **19** 1095–7

[2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 Nonlinear evolution equations of physical significance *Phys. Rev. Lett.* **31** 125–7

[3] Budinich P and Trautman A 1988 *The Spinorial Chessboard* (Berlin: Springer)

[4] Cieśliński J 1997 The cross ratio and Clifford algebras *Adv. Appl. Clifford Alg.* **7** 133–9

[5] Ablowitz M J, Beals R and Tenenblat K 1986 On the solution of the generalized wave and generalized sine–Gordon equations *Stud. Appl. Math.* **74** 177–203

[6] Terng C L 1980 A higher dimension generalization of the sine–Gordon equation and its soliton theory *Ann. Math.* **111** 491–510

[7] Lund F 1978 Classically solvable field theory model *Ann. Phys., NY* **115** 251–68

[8] Sym A 1982 Soliton surfaces *Lett. Nuovo Cimento* **33** 394–400

[9] Sym A 1985 Soliton surfaces and their application. Soliton geometry from spectral problems *Geometric Aspects of the Einstein Equations and Integrable Systems (Lecture Notes in Physics 239)* ed R Martini (Berlin: Springer) pp 154–231

[10] Cieśliński J 1997 A generalized formula for integrable classes of surfaces in Lie algebras *J. Math. Phys.* **38** 4255–72

[11] Bobenko A I 1994 Surfaces in terms of 2 by 2 matrices. Old and new integrable cases *Harmonic Maps and Integrable Systems (Aspects of Mathematics 23)* ed A P Fordy and J C Wood (Brunswick: Vieweg)

[12] Hu H S 1995 Solitons and differential geometry *Soliton Theory and Its Applications* (Berlin: Springer) pp 297–336

[13] Fokas A S and Gelfand I M 1996 Surfaces on Lie groups, on Lie algebras and their integrability *Commun. Math. Phys.* **177** 203–20

[14] Chern S S 1991 Surface theory with Darboux and Bianchi *Miscellanea Mathematica* (Berlin: Springer) pp 59–69

[15] Sym A 1983 Soliton surfaces II. Geometric unification of solvable nonlinearities *Lett. Nuovo Cimento* **36** 307–12

[16] Cieśliński J 1998 The Darboux–Bianchi–Bäcklund transformations and soliton surfaces *Nonlinearity and Geometry* ed D Wójcik and J Cieśliński (Warsaw: PWN)

[17] Cieśliński J, Gragert P K H and Sym A 1986 Exact solution to localized-induction-approximation equation modelling smoke ring motion *Phys. Rev. Lett.* **57** 1507–10
[18] Fukumoto Y and Miyazaki T 1991 Three-dimensional distortions of a vortex filament with axial flow *J. Fluid Mech.* **222** 369–416
[19] Konno K, Mituhashi M and Ichikawa Y M 1991 Soliton on thin vortex filament *Chaos Solitons Fractals* **1** 55–65
[20] Cieśliński J 1997 The spectral interpretaton of n-spaces of constant negative curvature immersed in R^{2n-1} *Phys. Lett.* A **236** 425–30
[21] Cieśliński J 2000 A class of linear spectral problems in Clifford algebras *Phys. Lett.* A **267** 251–5
[22] Cieśliński J 1997 The Darboux–Bianchi transformation for isothermic surfaces. Classical results versus the soliton approach *Diff. Geom. Appl.* **7** 1–28
[23] Tzarev S P 1992 Classical differential geometry and integrability of systems of hydrodynamic type *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations (NATO ASI Series)* ed P A Clarkson (Dordrecht: Kluwer) pp 241–9
[24] Aminov Yu A 1982 Multidimensional analogue of the sine–Gordon equation and the motion of a rigid body *Dokl. Akad. Nauk SSSR* **264** 1113–16 (in Russian) (Engl. Transl. 1982 *Sov. Phys.–Dokl.* **27**)
[25] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627–89